APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2022 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 25, 2022

Time: 9:00 AM – 1:00 PM

A1. Solve the initial value problem:

$$u_{tt} - u_{xx} = 0$$

$$u(x,0) = \begin{cases} \sin \pi x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}, \qquad u_t(x,0) = \begin{cases} \pi \cos \pi(x-2) & \text{if } 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}.$$

Find and draw the solutions at $t = 1, 2, 3$ if

(1). $x \in (-\infty, \infty)$.

(2). $x \in (0, \infty)$ with $u_x(0,t) = 0$.

For both (1) and (2)

(3). Find and draw the domain of dependence of the point $(x,t) = (1,2)$.

(4). Find and draw the range of influence of the point $(x,t) = (1,0)$ up to $t = 3$.

A2. Solve Laplace equation

$$\Delta u = 0, \quad \Omega : \{x \in (0,1), y \in (0,1)\}$$

with the boundary condition

$$u(x,0) = 0, \quad u(x,1) = 4x(1-x)$$

$$u(0,y) = \sin 2\pi y, \quad u(1,y) = \sin \pi y.$$

Find the maximum and minimum of the solution.

A3. For the conservation law

$$u_t + f(u)_x = 0,$$

where flux function is $f(u) = u(1+u)$.

$$u(x,0) = \begin{cases} u_l & x < -1 \\ u_m & -1 < x < 1 \\ u_r & x > 1 \end{cases}$$

(a). Draw characteristics and find solution $u(x,t)$ if $u_l = u_m = 1$ and $u_r = 3$.

(b). Draw characteristics and find solution $u(x,t)$ if $u_l = 3$ and $u_r = u_m = 1$.

(c). Draw characteristics and find solution $u(x,t)$ if $u_l = 3$, $u_m = 2$ and $u_r = 1$.

(d). Solve the Riemann problem

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

(e). Find the Riemann solution at $x/t = 0$.

A4. Prove for any $\alpha \in \mathbf{C}$ and $n \geq 2$ that the polynomial $\alpha z^n + z + 1$ has at least one root in the disc $\{|z| \leq 2\}$.

A5. Compute

$$I = \int_0^{\pi} \frac{1}{2 - \cos x} dx.$$

B6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\boldsymbol{x}) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2.$$

a) Use the first- and second-order optimality conditions to show that $\boldsymbol{x}^* = [1, 1]$ is a local minimum.

b) Give the linear system for the first iteration of Newton's method for minimizing f using $\boldsymbol{x}_0 = [2, 2]^T$ as the starting point.

c) Give the first iteration of the steepest descent method for minimizing f using $\boldsymbol{x}_0 = [2, 2]^T$ as the starting point.

B7. For an initial value problem

$$y'(t) = f(t, y),$$

with $y(0) = y_0$, Heun's method given by

$$y_{k+1} = y_k + \frac{h}{2}(k_1 + k_2),$$

where

$$\begin{aligned} k_1 &= f(t_k, y_k) \\ k_2 &= f(t_k + h_k, y_k + h_k k_1). \end{aligned}$$

a) Write the Butcher array of this method in the form of $\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array}$. Recall that the general form of an s-stage Runge-Kutta method can be given as

$$u_{k+1} = u_k + hF(t_n, u_n, h; f), \quad n \geq 0,$$

where

$$\begin{aligned} F(t_n, u_n, h; f) &= \sum_{i=1}^{s} b_i K_i, \\ K_i &= f(t_n + c_i h, u_n + h \sum_{j=1}^{s} a_{ij} K_j), \quad i = 1, 2, \ldots, s \end{aligned}$$

b) Show that this method is second-order accurate.

c) Consider the special case $f(t, y) = \lambda y$ with $\lambda \in \mathbb{R}^-$. Find the maximum h such that the method is stable.

d) How does Heun's method compare against the trapezoid method in terms of stability and stability?

B8. Consider the boundary value problem

$$\begin{aligned} -u''(x) + b(x)u(x) &= f(x),\ 0 \le x \le 1 \\ u(0) = u_l, \qquad & u(1) = u_r \end{aligned} \tag{1}$$

where $b(x)$, $f(x)$ are continuously differentiable, and $b(x) \ge 0$.

a) Formulate a second-order difference method for finding the approximate solution of (**??**) on a uniform mesh of size h.

b) Suppose $b(x) = 0$ and $u_l = u_r = 0$ in (a). Formulate a finite element method for finding the approximate solution of (**??**) in this special case, also on a uniform mesh. Using the standard "hat functions" basis for the finite element space, write out the equations for the Galerkin method.

B9.

(a) Describe two versions of the Godunov method for hyperbolic conservation laws and comment on which one has implementation advantages.

(b) Solve the Riemann problem for a linear system of strictly hyperbolic PDE's

$$U_t + AU_x = 0, \quad U(x,0) = \begin{cases} U_l, & x < 0 \\ U_r, & x > 0 \end{cases}$$

(i.e., express the Riemann problem solution in terms of eigenvalues, eigenvectors, and the jump $[U] = U_r - U_l$).

(c) Using the Riemann problem solution, derive the Godunov method for this linear system and show that it is equivalent to the upwind method.

B10.

(a) Derive an ADI scheme for the 2D parabolic equation $v_t = \nu(v_{xx} + v_{yy})$ by performing an incomplete factorization of the BTCS scheme.

(b) Discuss its consistency and perform stability analysis using the discrete Fourier transform.

(c) Discuss advantages and disadvantages of your scheme compared to the BTCS scheme and the Pieceman-Rachford ADI scheme.